

Nullstellensatz over quasifields*

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Abstract

We investigate the least studied class of differential rings – the class of differential rings of nonzero characteristic. We present the notion of differentially closed quasifield and develop geometrical theory of differential equations in nonzero characteristic. The notions of quasivarieties and their morphisms are scrutinized. Presented machinery is a basis for reduction modulo p for differential equations.

1 Introduction

We investigate the least studied class of differential rings – the class of differential rings of nonzero characteristic. Almost all known technique devoted to the differential rings is based on methods invented by Kolchin [11]. However, this machinery is not appropriate in the case when the set of all differential prime ideals is not big enough. So the rings with empty differential spectrum are out of the field of application of the known technique. There are a few number of early works of Yuan [17], Block [2], and Posner [13], [14]. Only at the end of 70s William Keigher obtained first new results about simple differential rings in nonzero characteristic. His latest work [10] is an attempt of generalization the Picard-Vessiot theory to the case of quasifield. In our work we present a slightly different direction: we shall generalize the algebraic geometry to the case of quasifield. However, it should be noted that we use Keigher's result as a background. Particular, the notion of quasifield and quasispectrum are key notions in our work. The problem of obtaining the results discussed was posed by Keigher in April, 2007. From his point of view this work should be a basis of the reduction modulo p for differential equations. To realize our plans we develop a new technique based on Nullstellensatz in the case of infinite number of indeterminates and on universal property of the Hurwitz series ring. The described method allows us not only to define the notions we interested in but scrutinize them well.

The paper consists of the main part and appendix. In sections 2 and 3 we present the basic terms and notations. In the section 2 we collect all terms and obvious facts. Also we give a references to the works where discussed notions investigate more accurate. In the section 3 we give two main statements which form a heart of our machinery, these are theorem 1 and 4. In spite of the proofs of these statements are very easy, we

*XY-pic package is used

call them theorems because they play very important role in our work. It should be noted that theorem 1 is just useful variant of theorems proved in appendix. In the section 4 we present the notion of differentially closed quasifield and scrutinize its structure. The main result of the section is a full classification of differentially closed quasifields. The section is ended by theorem 11 where we collect all proved about such quasifields facts. Section 5 is devoted to the notion of differential closure. We define it in the same way as it done for differential closure of the differential field of zero characteristic [12, chapter 6, sec. b]. Also we discuss the question of explicit construction of differential closure and the question of its minimality theorem 12 and 14, correspondingly. The ascending chain condition does not hold in the ring of differential polynomials over quasifields even for quasiradical ideals. Therefore section 6 is devoted to its weaker analogue, the notion of ω -Noetherian. The main theorem of this section is theorem 17. Sections from 7-th till 10-th are devoted to the geometric theory. The notion of differentially closed quasifield allows us to define the notion of quasivariety. Section 7 present affine quasivarieties and its basic properties. We do not define general variant of quasivariety but we assume that it can be done immediately by anyone. The section mostly devoted to the translation of proved facts to the geometric language. Additionally, we give some examples and discuss the relation between quasivariety and spectrum. To produce the category of quasivarieties we introduce the notion of its morphisms. For that in section 8 we define the notion of regular functions and mappings. Theorem 25 says that all regular functions are polynomial (compare with differential algebraic geometry [3, chapter I, sec. 5, p. 901]). The discussion is finished by categorical meaning of the proved result (theorem 27). In algebraic geometry using the notion of Noetherian, a lot of geometrical facts can be expressed in terms of constructible sets. Noetherian does not hold for quasivarieties. However, using more weaker notion of ω -Noetherian, we are able to prove some geometrical properties in terms of the Baire property. Section 9 is devoted to the Baire property. We discuss the behavior of irreducible quasivarieties under regular mappings. The last section devoted to the notion of quasivariety is section 10, in which the notion of dimension is introduced. The last three sections are devoted to some technical details and generalizations. From section 5 it follows that the structure of differential closure in the case of quasifield with countable residue field very differs from that in the contrary case. Theorem 33 in section 11 describe structure of differential closure in the hard case of countable residue field. The result shows that differential closure is constructible over basic quasifield (terms [12, chapter 10, sec. d]). In the section 12 we generalize the obtained results to the case of arbitrary set of differential indeterminates. The final section is appendix. This section is mostly devoted to the accurate proving of general form of Nullstellensatz.

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2 Terms and notation

Throughout the all text a differential ring A is a commutative, associative unitary ring with a finitely many pairwise commuting differential operators. The set of all differential operators will be denoted by Δ and the set of all derivations by Θ . Additionally, we assume that all ring are of nonzero characteristic. All homomorphisms are assumed to be unitary. Differential ring will be called simple if it has no non trivial differential ideals except zero ideal and whole ring. All terms and notation are the same as in [1] and [11]. If something is not defined or unexplained we mean it has the same meaning as in the mentioned sources.

The following terms mostly are introduced in [9] and [5]. We shall briefly recall them. An element x of a differential ring A is called a differential nilpotent if θx is nilpotent whenever $\theta \in \Theta$, in other words, the ideal $[x]$ belongs to the nilradical of the ring. An ideal \mathfrak{u} will be said to be quasiradical if a quotient ring A/\mathfrak{u} has no nonzero differential nilpotents. The ring with the last property will be called quasireduced. An ideal \mathfrak{q} will be said to be quasiprime if it is quasiradical and primary. An ideal \mathfrak{m} will be said to be quasimaximal if it is a maximal differential ideal.

Basic properties and relations between mentioned types of differential ideals are scrutinized in [9] and [5]. We recall just needed facts below. The sets of all radical, prime, maximal ideals of a ring A are denoted by $\text{Rad } A$, $\text{Spec } A$, $\text{Max } A$, respectively. In analogue way we shall denote by $\text{QRad } A$, $\text{QSpec } A$, $\text{QMax } A$ the set of all quasiradical, quasiprime, and quasimaximal ideals, respectively. It is easy to see that any quasiradical ideal can be presented as the intersection of all quasiprime ideals containing it. The least quasiradical ideal containing an ideal \mathfrak{a} will be denoted by $\text{rad}(\mathfrak{a})$. The last one coincides with the set of all differential nilpotents modulo \mathfrak{a} . In other words, the behavior of the ideals with prefix “quasi” is the same as that of the ideals without it.

Let define the following mappings on ideals:

$$\begin{aligned}\pi: \text{Rad } R &\rightarrow \text{QRad } R \\ \tau: \text{QRad } R &\rightarrow \text{Rad } R\end{aligned}$$

where $\pi(\mathfrak{t})$ is the largest differential ideal belonging to \mathfrak{t} and $\tau(\mathfrak{u})$ is a radical of the ideal \mathfrak{u} in the usual sense [1, chapter 1, sec. 6, p. 8]. The mapping π is scrutinized in [5], but with the following notation $\pi(\mathfrak{a}) = \mathfrak{a}_\Delta$. The mapping π preserve arbitrary intersections [5, prop. 1.1].

The set $\text{Spec } A$ is provided with Zarisky topology [1, chapter 1, ex. 15]. Let provide topology on $\text{QSpec } A$. The set $\text{QSpec } A$ with topology introduced below will be called a quasispectrum. For each subset $E \subseteq A$, let $V(E)$ denote the set of all quasiprime ideals of A which contain E . From statement 37 in appendix it follows that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. For each $x \in A$, let $(\text{QSpec } A)_x$ denote the complement of $V(x)$ in $\text{QSpec } A$. The mentioned sets are open and form a basis of open sets for the introduced topology. From now and

till the end we shall mean only these two mentioned topology on spectra and quasispectra.

From the definition of the mapping π it follows that π is continuous. In the case of arbitrary characteristic the mapping τ is not necessarily continuous, but in the case of nonzero characteristic the following fact holds: the mappings π and τ are inverse to each other bijections between the set of all radical ideals and the set of all quasiradical ideals. Moreover, these two mappings provide inverse to each other homeomorphisms between $\text{Spec } A$ ($\text{Max } A$) and $\text{QSpec } A$ ($\text{QMax } A$) (the prove is in [16, lemma 13 and theorem 14]). Let denote everything on the following diagram

$$\begin{array}{ccccc} \text{Max } R & \xrightarrow{\quad} & \text{Spec } R & \xrightarrow{\quad} & \text{Rad } R \\ \tau \updownarrow \pi & & \tau \updownarrow \pi & & \tau \updownarrow \pi \\ \text{QMax } R & \xrightarrow{\quad} & \text{QSpec } R & \xrightarrow{\quad} & \text{QRad } R \end{array}$$

It should be noted, that presented topology on $\text{QSpec } A$ coincides with topology introduced in [8, sec. 2, p. 144].

A ring A will be said to be a quasifield if it is simple differential ring of nonzero characteristic. From [17, sec. 2, theor. 2.1] it follows that A is a local ring of Krull dimension zero having prime characteristic p . Its maximal ideal is determined by the following condition

$$\mathfrak{m} = \{ x \in A \mid x^p = 0 \}.$$

The residue field of the ideal \mathfrak{m} will be denoted by $K = A/\mathfrak{m}$ and called a residue field of quasifield A . An additional information about quasifield structure can be found in [5, sec. 2, prop. 2.8].

3 Basic technique

The most popular technique for investigating differential rings is based on characteristic sets. However, this machinery does not provide desired results in nonzero characteristic. The section is devoted to the adequate ring machinery providing our needs.

The following result was found in [15, sec. 2, ex. 2.1, p. 18]. However, its original prove is based on algebras representation theory. We give the most useful version of the result. The prove of the full sequence of the general Hilbert Nullstellensatz is given in appendix.

Theorem 1. *Let $K \subseteq L$ be a field extension such that L is generated over K by not more than κ elements (κ is an arbitrary cardinal). Then, if $|K| > \kappa$ then L is algebraic over K .*

Proof. The proof follows from 38. □

From the result above the following one immediately follows.

Corollary 2. *Let K be an algebraically closed field and X be arbitrary set such that $|X| < |K|$. Then for each maximal ideal \mathfrak{m} of the ring $K[X]$ its quotient ring $K[X]/\mathfrak{m}$ coincides with K .*

Consider the example showing that the condition $|X| < |K|$ can not be omitted. This example is a partial case of theorem 38.

Example 3. Let K be a countable algebraically closed field and let L be algebraical closure of $K(x)$, where x is transcendental over K . It is easy to see that L is countable, and therefore there is a surjective homomorphism of the ring $K[x_n]_{n \in \mathbb{N}}$ onto L . The kernel of the last one is the desired example.

For any commutative ring A the ring of Hurwitz series is defined in [6, sec. 2]. We shall define it by HA . The Hurwitz series ring is a differential ring. From the described correspondence between spectrum and quasispectrum it follows that HA is a quasifield iff A is a field. The set of all series with zero free term will be denoted by HA_1 . In nonzero characteristic the ideal HA_1 consists of nilpotent elements. The mapping from HA to A , maps each series to its free term, is a ring homomorphism and is denoted by π . The following result is proved in [6, sec. 2, prop. 2.1] and [4, p. 100] (in the case of one derivation) and in [16, p. 216] (in generale case).

Theorem 4. *Let A be an arbitrary ring, B be a differential ring, and $\varphi: B \rightarrow A$ be a ring homomorphism. Then there exists a unique differential ring homomorphism Φ (the Taylor homomorphism) such that the following diagram is commutative*

$$\begin{array}{ccc} & & HA \\ & \nearrow \Phi & \downarrow \pi \\ B & \xrightarrow{\varphi} & A \end{array}$$

For any quasifield Q , its residue field K , and arbitrary set Y of differential indeterminates over Q the quotient ring of $Q\{Y\}$ by its nilradical will be denoted by $K\{Y\}$.

4 Differentially closed quasifields

We shall introduce the notion of differentially close quasifield and classify all such quasifields up to isomorphism.

Let Q be any quasifield. Consider the ring of differential polynomials $Q\{y_1, \dots, y_n\}$ over Q . For each subset $E \subseteq Q\{y_1, \dots, y_n\}$, let $V(E)$ denote the set of all common zeros in Q^n for the polynomials in E . Conversely, for each subset X of Q^n , let $I(X)$ denote the set of all polynomials vanishing on X . It is clear, that for any differential ideal \mathfrak{a} there is the inclusion $\text{rad}(\mathfrak{a}) \subseteq I(V(\mathfrak{a}))$.

A quasifield Q will be said to be differentially closed if for any natural number n and any differential ideal \mathfrak{a} of $Q\{y_1, \dots, y_n\}$ there is the equality $\text{rad}(\mathfrak{a}) = I(V(\mathfrak{a}))$.

Statement 5. *The following conditions on quasifield Q are equivalent:*

1. Q is differentially closed.
2. For any natural number n and any proper differential ideal \mathfrak{a} of $Q\{y_1, \dots, y_n\}$ the set $V(\mathfrak{a})$ is not empty.

3. Any differentially finitely generated over Q quasifield coincides with Q .

Proof. Implication (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3). Let L be differentially finitely generated over Q quasifield. Then L is of the form $Q\{y_1, \dots, y_n\}/\mathfrak{m}$ for some quasimaximal ideal \mathfrak{m} . By the data, the set $V(\mathfrak{m})$ contains a point $a \in Q^n$. Then $\mathfrak{m} \subseteq I(a)$ and hence $\mathfrak{m} = I(a)$, thus $L = Q$.

(3) \Rightarrow (1). Let \mathfrak{a} be a differential ideal of $Q\{y_1, \dots, y_n\}$ and let f is not in $\mathfrak{r}(\mathfrak{a})$. Then there exists a $\theta \in \Theta$ such that θf is not nilpotent modulo \mathfrak{a} . Therefore, algebra $B = Q\{y_1, \dots, y_n\}_{\theta f}/\mathfrak{a}$ is not zero. Consider a quasimaximal ideal \mathfrak{m} in B , then $B/\mathfrak{m} = Q$. The images of the elements y_1, \dots, y_n give us a point a in Q^n belonging to $V(\mathfrak{a})$ and such that $f(a) \neq 0$. \square

The following statements guaranties that differentially closed quasifields exist.

Statement 6. *Let K be more than countable algebraically closed field. Then HK is a differentially closed quasifield.*

Proof. Let show that the condition (3) of statement 5 is satisfied. Indeed, let L be a differentially finitely generated over HK quasifield. Then it can be presented in the following form

$$L = HK\{y_1, \dots, y_n\}/\mathfrak{m},$$

where \mathfrak{m} is a quasimaximal ideal. We shall construct a homomorphism from L to K . Let \mathfrak{n} be the radical of the ideal \mathfrak{m} . It is easy to see, that it contains the ideal $HK_1\{y_1, \dots, y_n\}$. Then from corollary 2 it follows that

$$HK\{y_1, \dots, y_n\}/\mathfrak{n} = K\{y_1, \dots, y_n\}/\mathfrak{n} = K.$$

Let denote the constructed homomorphism by φ . Applying theorem 4 we get that there exists a unique homomorphism Φ such that

$$\begin{array}{ccc} & & HK \\ & \nearrow \Phi & \downarrow \pi \\ L & \xrightarrow{\varphi} & K \end{array}$$

From the uniqueness of the Taylor homomorphism it follows that Φ is a homomorphism over HK . Since L is quasifield, the mapping Φ is an isomorphism over HK . \square

Example 3 implies the following result.

Statement 7. *Let Q be differentially closed quasifield and let K be its residue field. Then K is more than countable.*

Proof. Indeed, suppose that contrary holds. Consider the ring $K[\theta y]_{\theta \in \Theta}$ and its ideal \mathfrak{m} constructed as in example 3. The ideal can be extended to the ideal \mathfrak{m}' of $Q\{y\}$. Let put $\mathfrak{q} = \pi(\mathfrak{m}')$. We shall shaw that quasifield

$D = Q\{y\}/\mathfrak{q}$ does not coincide with Q . Let compare residue fields for that.

$$Q\{y\}/\mathfrak{m}' = K[\theta y]_{\theta \in \Theta}/\mathfrak{m} = \overline{K(x)}.$$

The contradiction finished the proof. \square

Statement 8. *Quasifield HK is differentially closed iff K is more than countable algebraically closed field.*

Proof. The right to left statement is proved in 7. The other side statement follows from the last one. \square

Statement 9. *Let Q be a quasifield with a residue field K . Suppose that any quasifield L , generated by one element over Q , coincides with Q . Then the Taylor homomorphism is an isomorphism between Q and HK .*

Proof. Let $\varphi: Q \rightarrow K$ be a quotient homomorphism and let $\Phi: Q \rightarrow HK$ be corresponding to φ the Taylor homomorphism. Since Q is a quasifield, we only need to check that Φ is surjective. Let suppose that contrary holds. In other words, there exists an element $\eta \in HK$ not belonging to Q . Let denote its coefficients by a_θ and consider the ideal $(\dots, \theta y - a_\theta, \dots)$ in $K\{y\}$. This ideal extends to maximal ideal \mathfrak{m} in $Q\{y\}$. Let \mathfrak{q} be corresponding to \mathfrak{m} quasimaximal ideal. Then, from the one hand, $Q\{y\}/\mathfrak{q}$ coincides with Q by the data. From the other hand, the image of $Q\{y\}/\mathfrak{q}$ under the Taylor homomorphism contains η , contradiction. \square

So, we have just classified all differentially closed quasifields. Let underline it in the following statement.

Statement 10. *Differentially closed quasifields are exactly the Hurwitz series rings over more than countable algebraically closed fields.*

Now we are able to prove generale result about differentially closed quasifields.

Theorem 11. *The following conditions on quasifield Q are equivalent:*

1. Q is differentially closed.
2. For any natural number n and any proper differential ideal \mathfrak{a} of $Q\{y_1, \dots, y_n\}$ the set $V(\mathfrak{a})$ is not empty.
3. Any differentially finitely generated over Q quasifield coincides with Q .
4. Any quasifield generated over Q by one single element coincides with Q .
5. Any quasifield generated over Q by not more than countable set of elements coincides with Q .
6. Q isomorphic to the Hurwitz series rings over more than countable algebraically closed fields.

Proof. Only one thing we need to show is that condition (5) follows from the others. From theorem 4 it follows that we need to show that for any such quasifield L there is a homomorphism from L to the residue field of Q . Let the residue field of Q be denoted by K . The residue field of L is not more than countably generated over K and thus from corollary 2 it coincides with K , q.e.d. \square

5 Differential closure

The following step is to define a differential closure for arbitrary quasifield. Let Q be a quasifield. We shall say that L a differential closure of Q if for any differentially closed quasifield D containing Q there is a homomorphism from L to D over Q . We shall prove that differential closure exists and unique up to isomorphism. Additionally we shall discuss the minimality question.

Let us describe the construction of differential closure. Let Q be a quasifield with a residue field K . If the field K is more than countable, then L will be defined as an algebraic closure of K . If the field K is not more than countable, then L will be defined as an algebraic closure of $K(x_\alpha)_{\alpha \in \omega_1}$, where ω_1 is the first more than countable cardinal. The differential closure \overline{Q} will be defined as $H L$. Moreover, we have the following sequence of homomorphisms

$$Q \rightarrow K \rightarrow L.$$

Applying the Taylor homomorphism to the last one, we get the inclusion of Q to its differential closure \overline{Q} .

Theorem 12. *Let Q be a quasifield with a residue field K . Then a differential closure of Q is unique up to isomorphism and is isomorphic to the quasifield \overline{Q} .*

Proof. The fact that \overline{Q} is differentially closed follows from statement 8. We shall prove the universal property. Let D be a differentially closed quasifield containing Q and let the residue field of D is denoted by F . Then K is embedded to F . Moreover, from statement 8 it follows that F is more than countable algebraically closed field. Therefore, the embedding K to F can be extended to embedding L to F (L is a residue field of \overline{Q}). Theorem 4 guaranties that there is an embedding \overline{Q} to D . From the uniqueness of the Taylor homomorphism it follows that this embedding is over Q . \square

Statement 13. *Let Q be a quasifield with a residue field K . If the field K is more than countable then for any differential ring D such that $Q \subseteq D \subseteq \overline{Q}$ it follows that D is a quasifield.*

Proof. To prove the result we just need to show that for each element $\eta \in \overline{Q}$ the ring $Q\{\eta\}$ is a quasifield. From the definition we have $\overline{Q} = H \overline{K}$. Consider the element $\eta \in H \overline{K}$ and let a_θ be its coefficients. Let define the ideal $(\dots, \theta y - a_\theta, \dots)$ in $\overline{K}\{y\}$. Since \overline{K} is integral over K , then the mentioned ideal contract to a maximal ideal \mathfrak{m}' in $K\{y\}$. The ideal \mathfrak{m}' extends to the maximal ideal \mathfrak{m} in $Q\{y\}$. Let $\mathfrak{q} = \pi(\mathfrak{m})$. Then, from one hand, $Q\{y\}/\mathfrak{q}$ is a quasifield. From the other hand, the ring $Q\{\eta\}$ coincides with the image of $Q\{y\}/\mathfrak{q}$ under the Taylor homomorphism. \square

Theorem 14. *Let Q be a quasifield with a residue field K . Differential closure \overline{Q} is minimal over Q iff the field K is more than countable.*

Proof. If the field K is more than countable then the desired result follows from the previous statement. Indeed, suppose that contrary holds, that there is a differentially closed quasifield D with condition $Q \subseteq D \subseteq \overline{Q}$. But for each element $y \in \overline{Q} \setminus D$ the algebra $D\{y\}$ is a quasifield and does not coincide with D , contradiction.

Let K be not more than countable. From the definition of the quasifield \overline{Q} it follows that there is a subfield L' in L containing K and isomorphic to L . Thus the quasifield $H L'$ is a differentially closed quasifield such that $Q \subsetneq H L' \subsetneq \overline{Q}$. \square

6 ω -Noetherian

We shall say that a particular ordered set S is ω -noetherian (ω is a countable cardinal) if each more than countable ascending chain is stable, or, in other words, every strictly ascending chain of elements is not more than countable. A ring A will be said to be ω -noetherian if the set of all its ideals is ω -noetherian. The following statement is obvious.

Statement 15. *A ring A is ω -noetherian iff each ideal of A is not more than countably generated.*

The main example of an ω -noetherian ring is a countably generated algebra over a field. It follows from the next result.

Statement 16. *The ring of polynomials $\mathbb{K}[x_n]_{n \in \mathbb{N}}$ is ω -noetherian.*

Proof. For any ideal \mathfrak{a} we have $\mathfrak{a} = \cup_n \mathfrak{a}_n$, where

$$\mathfrak{a}_n = \mathfrak{a} \cap K[x_1, \dots, x_n]$$

But each ideal \mathfrak{a}_n is finitely generated. Therefor, \mathfrak{a} is not more than countably generated. \square

The following fact is the analogue of Ritt-Raudenbusch's basis theorem.

Theorem 17. *Let Q be a quasifield with a residue field K . Then the set $\text{QRad } Q\{y_1, \dots, y_n\}$ is ω -noetherian.*

Proof. Let \mathfrak{n} be the nilradical of Q . Then the particular ordered sets

$$\text{QRad } Q\{y_1, \dots, y_n\} \text{ and } \text{Rad } Q\{y_1, \dots, y_n\}$$

are isomorphic. The last one is isomorphic to

$$\text{Rad } Q/\mathfrak{n}\{y_1, \dots, y_n\} = \text{Rad } K\{y_1, \dots, y_n\}.$$

Statement 16 finished the proof. \square

Corollary 18. *Each quasiradical ideal \mathfrak{u} of $Q\{y_1, \dots, y_n\}$ is not more than countably generated as a quasiradical ideal.*

7 Quasivarieties

We shall fix a differentially closed quasifield Q until the end of the text. Its residue field will be denoted by K . We shall deal with differential polynomial ring $Q\{y_1, \dots, y_n\}$ and its quotient by nilradical ring $K\{y_1, \dots, y_n\}$.

Statement 10 implies that

$$Q^n = (H K)^n = K^\Theta \times \dots \times K^\Theta.$$

Since $\text{QSpec } Q\{y_1, \dots, y_n\}$ is homeomorphic to $\text{Spec } K\{y_1, \dots, y_n\}$ we are able to identify the following topological spaces

$$Q^n = \text{QMax } Q\{y_1, \dots, y_n\} = \text{Max } K\{y_1, \dots, y_n\} = K^\Theta \times \dots \times K^\Theta,$$

where the element $(a_{\theta_1, 1}, \dots, a_{\theta_n, n})$ in $K^\Theta \times \dots \times K^\Theta$ corresponds to the element (q_1, \dots, q_n) in Q^n such that $a_{\theta_i, i}$ are the coefficients of the series q_i .

Consider an affine space Q^n . For each set of differential polynomials E in $Q\{y_1, \dots, y_n\}$, let define the following subset

$$V(E) = \{x \in Q^n \mid \forall g \in E: g(x) = 0\}.$$

A subset X in Q^n will be said to be an affine quasivariety (or simply quasivariety) if X is of the form $V(E)$ for some E . The sets $V(E)$ satisfy the axioms for closed sets in a topological space. The mentioned topology coincides with topology on spectrum under the bijection above.

Let X be an arbitrary subset in Q^n . Then we shall define the ideal

$$I(X) = \{f \in Q\{y_1, \dots, y_n\} \mid f|_X = 0\}.$$

From the definition of differentially closed quasifield the following result follows immediately.

Statement 19. *The mappings I and V are inverse to each other bijections between the set of all quasivarieties in Q^n and the set of all quasiradical ideals in $Q\{y_1, \dots, y_n\}$.*

It should be noted that in contrast to algebraic geometry (differential algebraic geometry) the set of all irreducible components of quasivariety is not necessarily finite.

Example 20. The notion of irreducible component is a topological notion. Since every quasivariety is homeomorphic to a spectrum of an algebra that is countably generated over a field, we just need to produce an example of such an algebra. Consider the polynomial ring $K[x_n]_{n \in \mathbb{N}}$ and the ideal

$$\mathfrak{a} = (\dots, x_i x_j, \dots)_{i \neq j}.$$

It is easy to see that the ideals

$$\mathfrak{p} = (\dots, x_j, \dots)_{j \neq l}$$

are its minimal prime components.

We shall call a topology space ω -noetherian if its every more than countable descending chain of closed sets is stable. In other words, statement 17 means the following.

Statement 21. *Every quasivariety is ω -noetherian.*

Another variant of the same result is the following.

Statement 22. *Every system of differential equations over quasifield is equivalent to its not more than countable subsystem.*

Any element of $Q\{y_1, \dots, y_n\}$ determines a differential polynomial function $Q^n \rightarrow Q$. Let X be a quasivariety in Q^n . A restriction of a differential polynomial mapping to X is called a differential polynomial function on X . The set of all differential polynomial functions on X will be denoted by $Q\{X\}$. This ring is isomorphic to $Q\{y_1, \dots, y_n\}/I(X)$. The quotient ring of $Q\{X\}$ by its nilradical will be denoted by $K\{X\}$. We know that X coincides with $\text{QMax } Q\{X\}$ and $\text{Max } K\{X\}$. A principal open set is a set of the following form

$$X_g = \{a \in X \mid g(a) \neq 0\}.$$

It is easy to see that homeomorphism between X and $\text{QMax } Q\{X\}$ preserves the notion of principal open. But it should be noted that this notion is not preserved under homeomorphism between $\text{QMax } Q\{X\}$ and $\text{Max } K\{X\}$. Namely, principal open set $(\text{Max } K\{X\})_g$ is mapped to principal open of the form $(\text{QMax } Q\{X\})_{g^p}$. The principal open sets of the form $(\text{QMax } Q\{X\})_{g^p}$ will be called a “good” principal open sets. As we can see, there is the equality

$$X_{g^p} = \text{QMax}(Q\{X\}_g).$$

Statement 23. *Let A be a ring of nonzero characteristic n . Then “good” principal open sets form a basis of topology on $\text{QSpec } A$ ($\text{QMax } A$).*

Proof. We shall prove it for quasispectrum. Let the following notation be fixed $X = \text{Spec } A$ and $Y = \text{QSpec } A$. Let $\pi: X \rightarrow Y$ be the mentioned homeomorphism. Consider

$$\begin{aligned} Y_f &= Y \setminus V([f]) = \pi(X \setminus V(\sum_k (f^{(k)}))) = \pi(X \setminus \bigcap_k V(f^{(k)})) = \\ &= \pi(\bigcup_k X_{f^{(k)}}) = \bigcup_k Y_{(f^{(k)})^n} \end{aligned}$$

Q.E.D. □

The geometric version of previous lemma is

Corollary 24. *The “good” principal open sets form a basis of topology on X .*

8 Regular functions and a structure sheaf

Let X be a quasivariety and let $f: X \rightarrow Q$ be a function on X . We shall say (compare with [8, sec. 3]) that f is regular in $x \in X$ if there are an open neighborhood U containing x and elements $h, g \in R$ such that for every element $y \in U$ $g(y)$ is not nilpotent and $f(y) = h(y)/g(y)$. The condition on g can be stated as follows: for each element $y \in U$

$g^p(y) \neq 0$. For any subset Y of X a function is said to be regular on Y if it is regular in each point of Y . The set of all regular functions on open subset U of X will be denoted by $\mathcal{O}_X(U)$. Since definition of regular function arises from local condition the set of rings \mathcal{O}_X form a sheaf that will be called a structure sheaf on X . This definition coincides with the usual one in algebraic geometry. From the definition it follows that there is the inclusion $Q\{X\} \subseteq \mathcal{O}_X(X)$. The very important fact that the other one is also true.

Theorem 25. *For arbitrary quasivariety X there is the equality*

$$Q\{X\} = \mathcal{O}_X(X).$$

Proof. Let f be a regular function on X . From the definition of regular function it follows that for each point $x \in X$ there exist a neighborhood U_x and elements $h_x, g_x \in Q\{X\}$ such that for every element $y \in U_x$ $g_x^p(y) \neq 0$ and $f(y) = h_x(y)/g_x(y)$. From corollary 24 it follows that each U_x can be presented as a union of “good” principal open sets and this family cover X . Since X is homeomorphic to a prime spectrum it follows that X is compact and thus there is a finite family of elements $\{a_i, b_i, c_i\}$ in $Q\{X\}$ such that $X_{c_i^p}$ cover X and for each $y \in X_{c_i^p}$ $b_i^p(y) \neq 0$ and $f(y) = a_i(y)/b_i(y)$. Since b_i is not nilpotent in $X_{c_i^p}$ we have $X_{(c_i b_i)^p}$ coincides with $X_{c_i^p}$. Thus we can suppose that there is a finite family of elements $\{a_i, b_i\}$ in $Q\{X\}$ such that $X_{b_i^p}$ cover X and for each $y \in X_{b_i^p}$ $b_i^p(y) \neq 0$ and

$$f(y) = \frac{a_i(y)}{b_i(y)} = \frac{a_i(y)b_i^{2p-1}(y)}{b_i^{2p}(y)} = \frac{a'_i(y)}{b'_i(y)},$$

where b'_i are constant and $a'_i(y) = 0$ whenever $b'_i(y) = 0$. Since X is covered by $X_{b'_i}$ we have $[b'_1, \dots, b'_n] = (b'_1, \dots, b'_n) = (1)$. Therefore

$$1 = h_1 b'_1 + \dots + h_n b'_n$$

And the element $r = h_1 a'_1 + \dots + h_n a'_n$ coincides with f everywhere on X . \square

The following result says that the structure sheaf is really a structure sheaf.

Statement 26. *For any open subset $U \subseteq X$ the mapping*

$$\text{QMax } \mathcal{O}_X(U) \rightarrow \text{QMax } Q\{X\}$$

is a homeomorphism onto its image U . In other words, there is a natural homeomorphism

$$\text{QMax } \mathcal{O}_X(U) = U.$$

Proof. The set U can be presented as a union of “good” open subsets $X_{g_i^p}$. Let denote $\text{QMax } \mathcal{O}_X(U)$ by Y . As we can see $Y = \cup_i Y_{g_i^p}$. It is suffices to show that for each i and j the mentioned mapping induces a homeomorphism between $X_{g_i^p} \cap X_{g_j^p}$ and $Y_{g_i^p} \cap Y_{g_j^p}$.

Indeed, consider a sequence of homomorphisms

$$Q\{X\} \rightarrow \mathcal{O}_X(U) \rightarrow Q\{X_{(g_i g_j)^p}\}.$$

Localizing by $(g_i g_j)$ we have

$$Q\{X\}_{g_i g_j} \rightarrow \mathcal{O}_X(U)_{g_i g_j} \rightarrow Q\{X\}_{g_i g_j}.$$

Everything that we need to check is that the second arrow in the last diagram is injective. But its kernel is a localization of that of $\mathcal{O}_X(U) \rightarrow Q\{X_{(g_i g_j)^p}\}$. The last one consists of all functions f that are zero on $X_{g_i^p} \cap X_{g_j^p}$ and thus satisfy the condition $(g_i g_j)^p f = 0$. \square

Let denote $\mathcal{O}_X(U)$ by $Q\{U\}$. Only fact needed to remember is that $Q\{U\}$ is not necessary differentially finitely generated over Q for arbitrary open U . And it should be noted that

$$Q\{X_f\} \neq Q\{X\}_f$$

for a nonconstant ($\delta f = 0$ for all $\delta \in \Delta$) element f (for example for any nonconstant nilpotent in Q).

Form theorem 25 it follows that a function coinciding with a restriction of some differential polynomial on X and a regular function are the same things. Now we shall define a morphisms. Let X and Y be quasivarieties such that $X \subseteq Q^n$ and $Y \subseteq Q^m$. The mapping $\varphi: X \rightarrow Y$ will be said to be a regular mapping if it can be defined by polynomials $\varphi_k(x_1, \dots, x_n)$ as follows

$$\varphi = (\varphi_1(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n))$$

It is clear that a regular mapping $\varphi: X \rightarrow Y$ defines a differential homomorphism

$$\varphi^*: Q\{Y\} \rightarrow Q\{X\}$$

by the formula $\xi \mapsto \xi \circ \varphi$. Conversely, any differential homomorphism $\phi: Q\{Y\} \rightarrow Q\{X\}$ define a regular mapping

$$\phi^*: \text{QMax } Q\{X\} \rightarrow \text{QMax } Q\{Y\}$$

by the rule $\phi^*(q) = \phi^{-1}(q)$, i.e. $\phi^*: X \rightarrow Y$. The last one implies the following result.

Theorem 27. *There is a contravariant equivalence between the category of quasireduced differentially finitely generated algebras over Q and the category of quasivarieties over Q .*

9 The Baire property

In algebraic geometry properties of the morphisms can be explained in terms of constructible sets. The section is devoted to an adequate analogue of this notion. We shall demonstrate the technique analogue to the Noether's normalization and derive from that different geometrical applications.

We shall say that the Baire property holds for a topological space X if the intersection of any countable family of dense open subsets is not empty. A subset is said to be big enough if it contains a mentioned intersection.

Lemma 28. *A “good” principal open set X_{f^p} is dense in X iff f is not a zero-divisor in $K\{X\}$.*

Proof. The proof immediately follows from the fact that $\text{Max } A_f$ is dense iff f is not zero-divisor in A . \square

Theorem 29. *The Baire property holds for irreducible quasivarieties.*

Proof. In irreducible variety any non empty open subset is dense. Since “good” principal open subsets form a basis of topology, we only need to prove the fact for “good” principal open subsets.

Let $X_{(s_n)^p}$ be a family of “good” principal open sets. From the previous statement it follows that s_n is not a zero divisor in $K\{X\}$. Let S be a multiplicative set generated by all elements s_n . Then algebra $S^{-1}Q\{X\}$ be nonzero and differentially not more then countably generated over Q . Therefor, from theorem 11 item (5) it follows that for any quasimaximal ideal \mathfrak{m} in $S^{-1}Q\{X\}$ there is the equality

$$S^{-1}Q\{X\}/\mathfrak{m} = Q.$$

Thus $Q\text{Max } S^{-1}Q\{X\}$ corresponds to the intersection $\cap_n X_{(s_n)^p}$ and is not empty. \square

The following result will be proved using the same ideas.

Theorem 30. *Let $\varphi: X \rightarrow Y$ be a regular mapping with a dense image and Y is irreducible. Then the image φ is a big enough subset in Y .*

Proof. Since image of X is dense in Y the algebra $Q\{Y\}$ is embedded in $Q\{X\}$. From the correspondence between maximal spectrum and quasimaximal spectrum it follows that we need to prove the statement for the pair $K\{Y\} \subseteq K\{X\}$. The algebra $K\{X\}$ is not more than countable union of finitely generated algebras over $K\{Y\}$, in other words

$$K\{X\} = \bigcup_{n=0}^{\infty} A_k,$$

where A_k is finitely generated over $K\{Y\}$. For each k there is s_k in $K\{Y\}$ such that the mapping $\text{Max}(A_k)_{s_k} \rightarrow \text{Max } K\{Y\}_{s_k}$ is surjective ([1, chapter 5, ex. 21] is a some variation on the Neother’s normalization theme). Let S be a multiplicative set generated by all elements s_k . Then the mapping

$$\text{Max } S^{-1}K\{X\} \rightarrow \text{Max } S^{-1}K\{Y\}$$

is surjective and the set $\text{Max } S^{-1}K\{Y\}$ corresponds to $\cap_k Y_{s_k^p}$ and belongs to image of φ . \square

10 Dimension

There is a standard technique of measuring a filtered algebra. We shall apply this machinery to the case of quasivariety.

Let X be a quasivariety. Consider the ring $W = K\{X\}$. The ring W is generated by the elements θy_k . We can define the subalgebra

$$W_r = K[\theta_1 y_1, \dots, \theta_n y_n \mid \text{ord } \theta_k \leq r]$$

and the function $\omega_X(t) = \dim W_t$ (Krull dimension). Additionally, we consider the following equivalence relation

$$f(n) \sim g(n) \Leftrightarrow f(n) \leq g(n + n_0) \quad g(m) \leq f(m + m_0)$$

Statement 31. *The equivalence class of the function ω_X does not depend on the choice of differential generators.*

Proof. Let W_r and W'_r be two different filtrations on W generated by two families of generators. Since W is covered by both sequences of subalgebras W_r and W'_r there are two numbers m_0 and n_0 such that $W_r \subseteq W'_{r+m_0}$ and $W'_r \subseteq W_{r+n_0}$. \square

The mentioned equivalence class we shall call a dimension of quasivariety. It should be noted that $\omega_X(t)$ may not coincide with polynomial function even for sufficiently large t .

11 Well-ordering on differential closure

Now let Q be an arbitrary quasifield with a residue field K . It is clear that statement 13 does not hold in the case of not more than countable residue field. For example, the residue field of \overline{Q} is not algebraic over K , and thus there is a subring in the residue field of \overline{Q} that is not a field. We shall show that even in this case there is some weaker analogue of statement 13 holds.

Statement 32. *Let L_1 and L_2 be quasifields containing Q . Then*

$$L_1 \otimes_Q L_2 \neq 0.$$

Proof. Let K_1 and K_2 be the residue fields of L_1 and L_2 respectively. Then the desired result immediately follows from the following

$$L_1 \otimes_Q L_2 \rightarrow K_1 \otimes_K K_2 \neq 0.$$

\square

Theorem 33. *Let Q be a quasifield with a residue field K . Then there is a well-ordered chain $\{L_\alpha\}$ of quasisubfields of \overline{Q} such that (i) $L_0 = Q$, (ii) $L_{\alpha+1}$ is generated by one single element over L_α , (iii) $L_\beta = \cup_{\alpha < \beta} L_\alpha$ for limit ordinals.*

Proof. We shall organize the proof in the following manner. Firstly, we shall construct a quasifield with desired properties and, secondary, we shall show, that the constructed quasifield coincides with \overline{Q} .

Let $\{B_\alpha\}_{\alpha \in \Lambda}$ be the family of all differentially finitely generated quasifields over Q up to isomorphism. We may suppose that Λ is well-ordered. Using induction by α we shall construct a sequence of quasifields L_α^1 . The quasifield L_0^1 will be defined as Q . For any limit ordinal α we have

$$L_\alpha^1 = \cup_{\beta < \alpha} L_\beta^1$$

and

$$L_{\alpha+1}^1 = (L_\alpha^1 \otimes_Q B_{\alpha+1})/\mathfrak{m},$$

for some quasimaximal ideal \mathfrak{m} . At the last step we shall construct a quasifield L^1 . This quasifield satisfies the following two properties: any homomorphism of Q to any differentially closed quasifield can be extended to a homomorphism of L^1 and for any differential ideal in $Q\{y_1, \dots, y_n\}$ there is a common zero in L^1 .

Using the same methods we produce the sequence of quasifields L^n . Consider L^{ω_1} , where ω_1 is the first more than countable ordinal. We shall show, that $L = L^{\omega_1}$ is the desired quasifield. Indeed, it suffices to show that L is differentially closed. Let \mathfrak{u} be a quasiradical ideal in $L\{y_1, \dots, y_n\}$, then from corollary 18 it follows that the ideal $\mathfrak{u} = \{r_n\}_{n \in \mathbb{N}}$ is not more than countable generated. Thus all r_n are determined over some L^β . Hence, there is a common zero in $L^{\beta+1}$ for all r_n . Since the kernel of the homomorphism

$$L\{y_1, \dots, y_n\} \rightarrow L,$$

where y_i is mapped to the coordinates of common zero, is a quasiprime ideal. Then whole ideal \mathfrak{u} is vanishing on the mentioned common zero. In other words, we have just checked the item (2) of theorem 11. \square

It should be noted that in the case of more than countable residue field the existence of the such sequence of the quasifield immediately follows from statement 13.

12 κ -closeness

Let Q be a quasifield with a residue field K . Let fix an infinite cardinal number κ . Consider the ring of differential polynomials $Q\{Y\}$ where Y is an arbitrary set. For any set of differential polynomials E the set of all its common zeros in Q^Y will be denoted by $V(E)$, i. e.

$$V(E) = \{a \in Q^Y \mid \forall g \in E : g(a) = 0\}.$$

Conversely, for any subset X in Q^Y the set of all polynomials vanishing on X is denoted by $I(X)$, i. e.

$$I(X) = \{f \in Q\{Y\} \mid f|_X = 0\}.$$

It is obvious that for any differential ideal \mathfrak{a} we have $\text{rad}(\mathfrak{a}) \subseteq I(V(\mathfrak{a}))$.

The quasifield Q will be said to be a κ -differentially closed if for any set Y such that $|Y| \leq \kappa$ and any differential ideal \mathfrak{a} in $Q\{Y\}$ there is the equality $\mathbf{rad}(\mathfrak{a}) = I(V(\mathfrak{a}))$.

The following theorem is the natural generalization of result 5 in the case of arbitrary number of variables.

Theorem 34. *The following conditions on quasifield Q are equivalent:*

1. Q is κ -differentially closed.
2. For any set X such that $|X| \leq \kappa$ and any proper differential ideal \mathfrak{a} in $Q\{X\}$ there is a common zero of \mathfrak{a} in Q^X .
3. Every quasifield generated by not more than κ elements over Q coincides with Q .

Proof. It is clear that the conditions (2) and (3) are equivalent. Also we see that (1) implies (2). Thus we just only need to show another implication. Let \mathfrak{a} be a differential ideal in $Q\{X\}$ and f is not in $\mathbf{rad}(\mathfrak{a})$. Then there is a derivative θf that is not a nilpotent modulo \mathfrak{a} . Thus the algebra $Q\{X\}_{\theta f}/\mathfrak{a}$ is not trivial and therefore the ideal $[\mathfrak{a}, z\theta f - 1]$ in $Q\{X \cup \{z\}\}$ is proper. Since κ is infinite then $|X| = |X \cup \{z\}|$ and from (2) it follows that f is not in $I(V(\mathfrak{a}))$. \square

In the same manner as statement 6 and theorem 9 the following result is proved.

Theorem 35. *A quasifield is κ -differentially closed iff it is isomorphic to a Hurwitz series ring over algebraically closed field of cardinality more than κ .*

Proof. Let show that the mentioned ring of Hurwitz series satisfies the mentioned properties. Let K be the mentioned algebraically closed field. Let show that condition (3) of theorem 34 holds.

Indeed, let L be differentially κ generated quasifield over HK . The it can be presented in the following form

$$L = HK\{Y\}/\mathfrak{m},$$

where \mathfrak{m} is a quasimaximal ideal and Y is of the cardinality less then or equal to κ . We shall construct a homomorphism from L to K . Let \mathfrak{n} be a radical of \mathfrak{m} , then it is clear that it contains $HK_1\{Y\}$. From theorem 40 it follows that

$$HK\{Y\}/\mathfrak{n} = K\{Y\}/\mathfrak{n} = K.$$

Let denote the constructed homomorphism $L \rightarrow K$ by φ . Now from theorem 4 it follows that there is a unique differential homomorphism Φ such that

$$\begin{array}{ccc} & HK & \\ & \downarrow \pi & \\ L & \xrightarrow{\varphi} & K \end{array} \quad \begin{array}{c} \nearrow \Phi \\ \searrow \end{array}$$

From the uniqueness of the Taylor homomorphism it follows that it defines over HK . Since L is a quasifield then Φ is an isomorphism.

Let us show the other implication. Let Q be κ -differentially closed quasifield with a residue field K . It suffices to show that K is of cardinality more than κ . Indeed, if it is so, the desired result immediately follows from statement 9.

Let suppose that contrary holds, in other words, that K is of cardinality less than or equal to κ . Then in the polynomial ring $K[\theta Y]$ there is a maximal ideal \mathfrak{m} such that its residue field does not coincide with K (see the proof of theorem 38). The ideal \mathfrak{m} can be extended to the maximal ideal \mathfrak{m}' in $Q\{Y\}$. Let $\mathfrak{q} = \pi(\mathfrak{m}')$ be a corresponding quasimaximal ideal. Let us show that quasifield $D = Q\{Y\}/\mathfrak{q}$ does not coincide with Q . For that we shall compare its residue fields

$$Q\{Y\}/\mathfrak{m}' = K[\theta Y]/\mathfrak{m} = L.$$

□

We are able to define the notion of a κ -noetherian particular ordered set: a particular ordered set is called κ -noetherian if any ascending chain of its elements of cardinality more than κ is stable. Following the proof of theorem 17 we get the following:

Theorem 36. *Let Q be an arbitrary quasifield and cardinality of Y is less than or equal to κ . Then the set $\text{QRad } Q\{Y\}$ is κ -noetherian.*

Proof. Let the residue field of Q is denoted by K . Let \mathfrak{n} be a nilradical of Q . We know that the particular ordered sets

$$\text{QRad } Q\{Y\} \text{ and } \text{Rad } Q\{Y\}$$

are isomorphic. The last one is isomorphic to

$$\text{Rad } Q/\mathfrak{n}\{Y\} = \text{Rad } K\{Y\}.$$

We shall show that the set of all ideals in the last ring is κ -noetherian. Indeed, Let Y be well-ordered and let Y_α be the subset of all elements with number less than or equal to α . Then every ideal \mathfrak{a} can be presented in the following form

$$\mathfrak{a} = \bigcup_{\alpha \leq \kappa} \mathfrak{a}_\alpha,$$

where $\mathfrak{a}_\alpha = \mathfrak{a} \cap K\{Y_\alpha\}$. From the well-known properties of cardinal numbers it follows that the ideal each ideal \mathfrak{a}_α is not more than α generated and thus \mathfrak{a} is not more than κ generated. □

It should be noted that the notion of κ -differentially closed quasifield allow us to define the notion of quasivariety in the affine space of any cardinal dimension and all results about usual quasivarieties can be generalized in to the case of arbitrary cardinal κ . But we shall not do it because it is a simple technical exercise and this question is out of our interests.

A Appendix

A.1 Topology on quasispectrum

Let A be an arbitrary differential ring, $X = \text{QSpec } A$ be its quasispectrum, and for each subset E of A we shall define the set

$$V(E) = \{ \mathfrak{q} \in \text{QSpec } A \mid E \subseteq \mathfrak{q} \}.$$

Statement 37. *In the mentioned notation the following statements holds*

1. *If \mathfrak{a} is a differential ideal generated by the set E then*

$$V(E) = V(\mathfrak{a}) = V(\text{rad}(\mathfrak{a})).$$

2. $V(0) = X$, $V(1) = \emptyset$.

3. *Let $\{E_i\}_{i \in I}$ be a family of subsets in A . Then*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

4. $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any differential ideals \mathfrak{a} , \mathfrak{b} in A .

Proof. To prove item (1) it suffices to show the following equality

$$\text{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{a} \subseteq \mathfrak{q}} \mathfrak{q},$$

where in the right part all ideals are quasiprime. Items (2) and (3) are obvious. Let show that item (4) holds. For that it suffices to show that any quasiprime ideal satisfies the following property: from inclusion $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$ it follows either $\mathfrak{a} \subseteq \mathfrak{q}$, or $\mathfrak{b} \subseteq \mathfrak{q}$. Indeed, we have the following sequence of inclusions

$$\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q} \subseteq \text{r}(\mathfrak{q}).$$

But $\text{r}(\mathfrak{q})$ is prime and thus, for example, $\mathfrak{a} \subseteq \text{r}(\mathfrak{q})$. But \mathfrak{a} is differential, therefore $\mathfrak{a} \subseteq \pi(\text{r}(\mathfrak{q})) = \mathfrak{q}$. \square

A.2 General forms of Nullstellensatz

Theorem 38 (Weak form, algebraic variant). *Let K be a field and κ is arbitrary cardinal, then the following are equivalent:*

1. $|K| > \kappa$ or κ is finite.
2. *Any overfield over K generated by not more than κ elements is algebraic over K .*

Proof. (1) \Rightarrow (2). If κ is finite then it is a well-known the Gilbert theorem. Let κ be infinite. Let X be the set of generators of L over K . Suppose that contrary holds. Let $Z \subseteq X$ be a transcendence basis of L over K , then $Y = X \setminus Z$ is algebraic over $K[Z]$. For any element $y \in Y$ there is a polynomial of algebraic dependance over $K[Z]$. Let its leading coefficient

is denoted by s_y . Let S be a multiplicative set generated by all elements s_y . Then L is integral over $S^{-1}K[Z]$. Hence from theorem [1, chapter 5, sec. 5, theor. 5.10] it follows that $S^{-1}K[Z]$ is a field too. Let fix one element $z \in Z$, then the set of polynomials $z - \alpha$, where $\alpha \in K$, is of cardinality more than κ . The set S is of cardinality less then or equal to κ , therefore for some α the polynomial $z - \alpha$ does not divide the elements od S . Thus $(z - \alpha)$ is nontrivial ideal of $S^{-1}K[Z]$, contradiction.

(2) \Rightarrow (1). Let suppose that $|K| \leq \kappa$ and κ is infinite. Then $L = K(x)$, where x is algebraically independent over K , is of cardinality less then or equal to κ . Therefore L is not more than κ generated over K . \square

Theorem 39 (Full form, algebraic variant). *Let K be a field, Y be an arbitrary set of cardinality κ , then the following are equivalent:*

1. $|K| > \kappa$ or κ is finite.
2. A polynomial ring $K[Y]$ is a Jacobson ring.

Proof. (1) \Rightarrow (2). If κ is finite then it is a well-known the Gilbert theorem. Let κ be infinite, let \mathfrak{a} be an ideal of $K[Y]$, and an element x does not belong to \mathfrak{a} . We need to show that there exists a maximal ideal \mathfrak{m} containing \mathfrak{a} and not containing x . From the choice of the element x it follows that the ring

$$K[Y]_x/\mathfrak{a}$$

is nontrivial. Let \mathfrak{m}' be its maximal ideal. Let its contraction to $K[Y]$ be denoted by \mathfrak{m} . We need to show that the ideal \mathfrak{m} is maximal too. Consider the following sequence of rings

$$K \subseteq K[Y]/\mathfrak{m} \subseteq K[Y]_x/\mathfrak{m}'.$$

From the previous theorem it follows that the last ring is integral over K . Hence the ring $K[Y]/\mathfrak{m}$ is integral over K . But K is a field, from theorem [1, chapter 5, sec. 2, prop. 5.7] it follows that $K[Y]/\mathfrak{m}$ is a field too.

(2) \Rightarrow (1). Item (2) implies the item (2) of the previous theorem, thus the first item holds too. \square

Theorem 40 (Weak form, geometric variant). *Let K be a field, Y be an arbitrary set of cardinality κ , then the following are equivalent:*

1. K is algebraically closed and $|K| > \kappa$.
2. For each nontrivial ideal \mathfrak{a} of $K[Y]$ there is a common zero in K^Y .

Proof. This result immediately follows from algebraic variant because algebraically closed field has no nontrivial algebraic extensions. \square

Theorem 41 (Full form, geometric variant). *Let K be a field, Y be an arbitrary set of cardinality κ , then the following are equivalent:*

1. K is algebraically closed and $|K| > \kappa$.
2. For each ideal \mathfrak{a} the following holds: $\mathfrak{r}(\mathfrak{a}) = I(V(\mathfrak{a}))$, where $V(\mathfrak{a})$ is the set of all common zeros of \mathfrak{a} in K^Y , $I(X)$ is the set of all polynomials vanishing on X , and $\mathfrak{r}(\mathfrak{a})$ is a radical.

Proof. From the previous statement it follows that the maximal ideals of $K[Y]$ corresponds to the points of K^Y . Thus our theorem follows from theorem 39. \square

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